Spontaneously broken symmetries
(in condensed matter, and in quantum magnets in particular)

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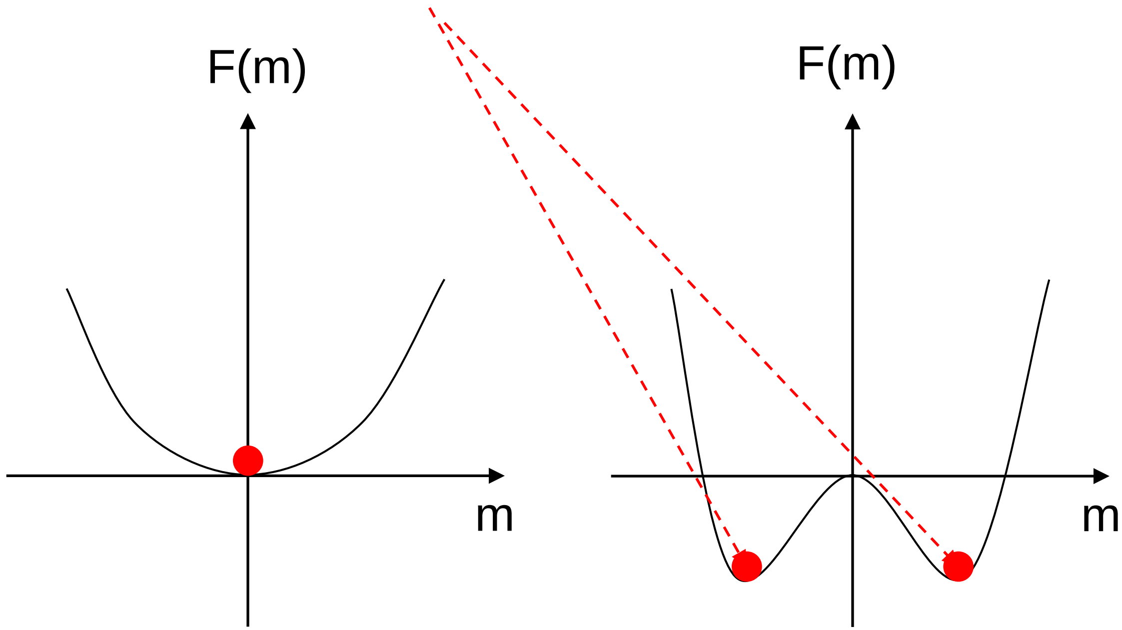
Broken symmetries

Uniaxial ferromagnet **m**: magnetization

Free energy F(m) has a **$m \Leftrightarrow -m$ symmetry**

but the values **$\pm m_0$** which minimizes F **breaks** this symmetry.

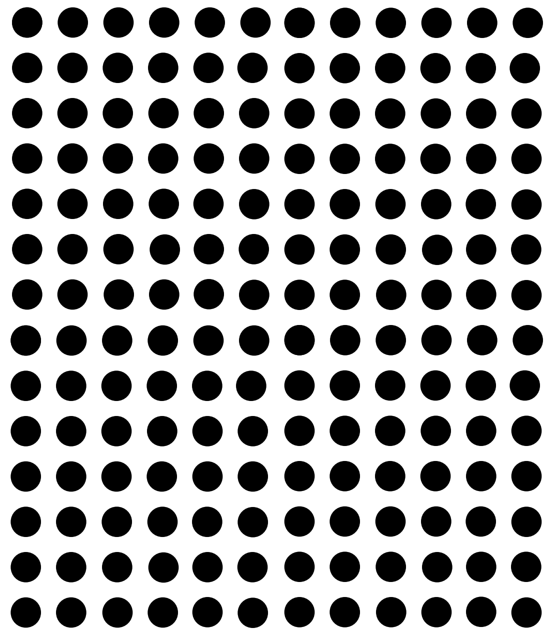


T>Tc
symmetric phase

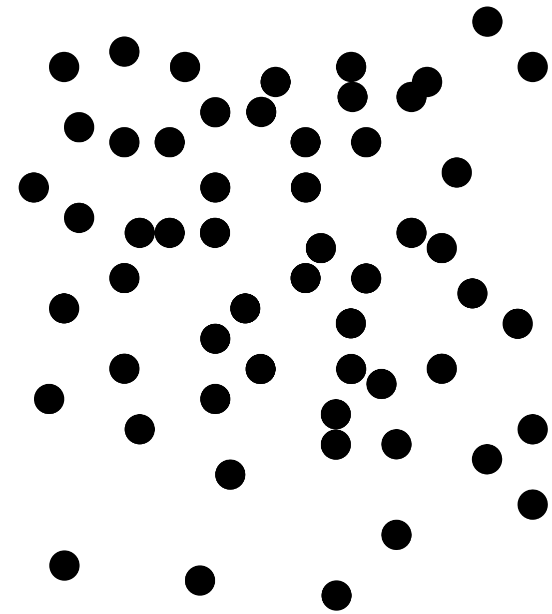
T<Tc
(spontaneously) **broken symmetry** phase
2-fold degenerate minimum

At T=Tc → *singularity* in the free energy (exercice using $F(m)=a(T-T_c)m^2+m^4$)

Which one is the most symmetric ?



Solid



Liquid

- A “snapshot” of the solid looks more symmetric
- But… a statistical ensemble, **the liquid is more symmetric**
- Example: the average particle density $n(r)$ is spatially uniform in the liquid, not in the solid
- The less symmetric phase (i.e. the solid) has some long-ranged order

Plan

- Introduction
  - Modèles, Hamiltoniens et symétries, définitions
  - Exemples simples de symétries brisées en physique statistique classique et quantique

- Paramètres d'ordre
  - définition(s)
  - exemples (et contre exemples!)
  - Un tout petit peu de théorie des groupes (& représentations)
  - Fonctions de corrélation, ordre à longue portée, susceptibilités
  - Théorie de Landau

- Brisure spontanée de symétries continues
  - Modes de Goldstone
  - Théorème de Mermin-Wagner:
  - Invariance de Jauge & mécanisme de Higgs

- Systèmes de taille finie
  Signature dans le spectre d'une brisure de sym., nombres quantiques, etc.

Models and symmetries, examples

Notations

- H: Hamiltonian (a priori quantum, but may be classical too)
- G: symmetry group.

- Group elements **act on states** g |i>=|g(i)> (unitary $g^{-1}=g^{+}$)

- Equivalently, group elements **act on operators/observables**: $O \to O' = g^{+} O g$

$$|a\rangle \to |a'\rangle = g|a\rangle$$
$$|b\rangle \to |b'\rangle = g|b\rangle$$

$$\langle a|O|b\rangle \to \langle a'|O|b'\rangle = \langle a|g^{+} O g|b\rangle$$

- g is a symmetry of H ⬄ $g^{-1}Hg = H$ ⬄ [g,H]=0

Symmetries - simplest examples

- Example1: **spin & rotations**

$$H=\sum_{\langle i,j\rangle} J_{ij}\vec{S}_i\cdot\vec{S}_i \quad \text{Heisenberg model}$$

$$g(\vec{n},\theta) = \text{global rotation of angle } \theta \text{ and axis } \vec{n}$$

$$=\exp\left(i\theta\sum_i \left(S_i^x n^x + S_i^y n^y + S_i^z n^z\right)\right)$$

$$\left[H,\sum_i S_i^\alpha\right]=0 \quad \Rightarrow \quad \left[H, g(\vec{n},\theta)\right]=0$$

- Example 2: **Atoms in a solid**.

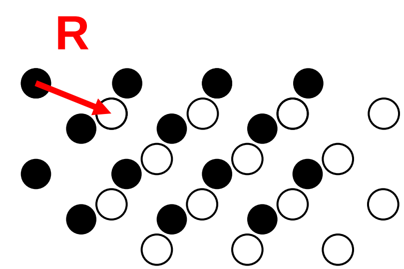
$$H=\sum_i \frac{\vec{p}_i^{\,2}}{2m}+\sum_{i<j} V(\vec{r}_i-\vec{r}_j)$$

Translation g_R : shifts the particle positions $r_i \rightarrow r_i + R$;

Corresponding operator:
[proof: check on plane waves]

$$g_{\mathbf{R}}=\exp\left(i\mathbf{R}\cdot\sum_j \mathbf{p}_j\right)$$

$\mathbf{r}_i \text{->} \mathbf{r}_i+\mathbf{R}$ does not change H ⇔ $P=\Sigma_j p_j$ is conserved]



- Solid state is *not* invariant under $r_i \rightarrow r_i + R$, contrary to liquids.

Classical Ising model: Z_2 sym. breaking & thermodynamic limit

- Ising model

$$E(\{\sigma\}) = -\sum_{\langle ij\rangle} \sigma_i \sigma_j \ , \quad \sigma_i = \pm 1$$

$$Z = \sum_{\sigma_i = \pm 1} \exp\left(- \frac{E(\{\sigma\})}{k_B T} \right)$$

$$\sigma_i \to -\sigma_i \quad \text{is a symmetry}$$

- Spontaneously broken in the low-temperature phase (d≥2): $T \geq T_c : \langle \sigma_i \rangle = 0$

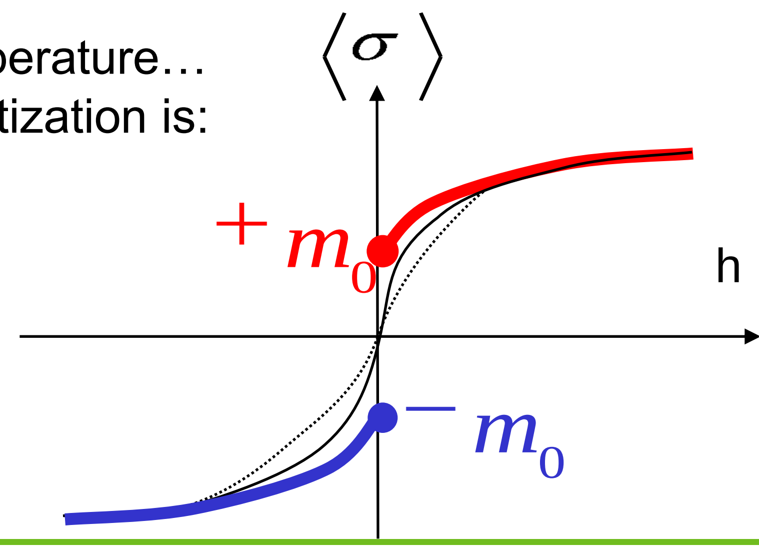
$$T < T_c : \langle \sigma_i \rangle = \pm m_0(T)$$

- Warning: **thermodynamic limit required !**

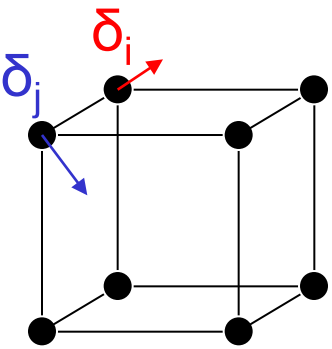
If the number of spins is finite → <σ>=0 at *any* temperature…
The proper way to measure a "spontaneous" magnetization is:

$$E = -\sum_{\langle ij\rangle} \sigma_i \sigma_j \underbrace{- h \sum_i \sigma_i}_{\text{ext. magnetic field}}$$

$$\langle \sigma_i \rangle = \lim_{h \to 0^+} \lim_{N \to \infty} \langle \sigma_i \rangle_{T,N,h}$$



Jahn-Teller distortion



Describes the atoms positions in a solid in terms of the deviation from their (high-temperature) equilibrium positions, which are assumed to form a regular (say cubic) lattice

$$H = \sum_{\langle i,j \rangle} V(\delta_i - \delta_j)$$

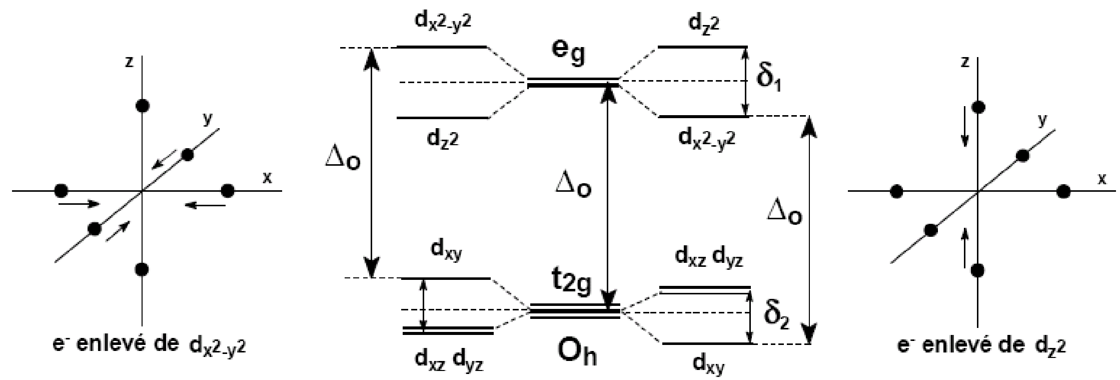
3 spatial directions are equivalent

V: complicated…:

-electrostatic interactions between electronic clouds

-electron kinetic energies

Electronic configuration & 3d orbitals



Spontaneous selection of one particular direction (driven by electronic energy gain)

Reduction of the lattice symmetries

Bose-Einstein condensation (bosons)

- Bose-Hubbard model

$$H = -t\sum_{\langle ij\rangle}\left(b_i^+ b_j + b_j^+ b_i\right) - \mu\sum_i b_i^+ b_i + U\sum_i b_i^+ b_i \left(b_i^+ b_i - 1\right)$$

- Bose condensation: non-zero expectation value of the creation/annihilation operator associated to the condensed (often k=0) mode

$$\langle b_{k_0}^+\rangle = \sqrt{Nn_c}\,\exp(i\varphi) \qquad \langle b_{k_0}^+ b_{k_0}^+\rangle = Nn_c$$

- φ = "phase of the condensate". Spontaneous break down of the U(1) symmetry
- But … what is the symmetry g_φ which rotates the phase φ ?
  Looking for g which satisfies $g_\varphi^{-1} b_i^+ g_\varphi = e^{i\varphi} b_i^+$

Operator which changes the phase : $g(\varphi) = \exp\left(i\varphi\sum_i b_i^+ b_i\right)$

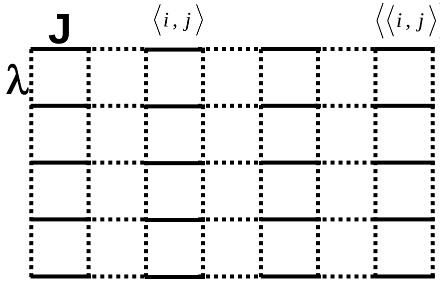
Particle conservation. $\left[H, g(\varphi)\right] = 0$

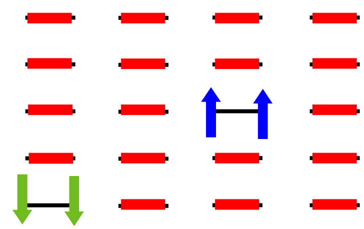
- What is the difference with the previous examples ?
  φ cannot be observed directly. It is "immaterial".

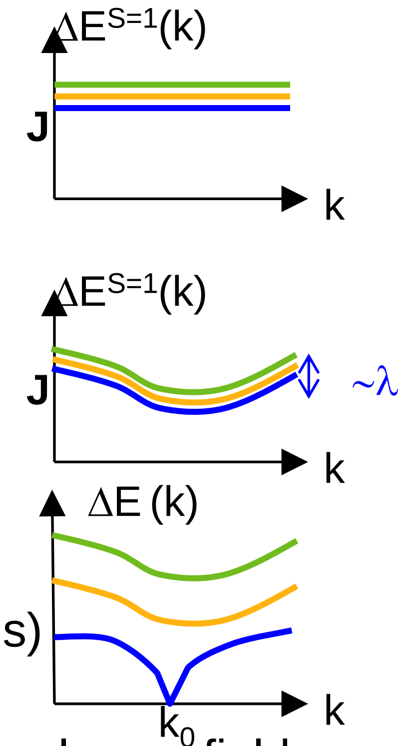
Bose-Einstein condensation of magnons

- Spin-½ Heisenberg model on a lattice made of coupled « dimers »

$$H = J\sum_{\langle i,j\rangle} \vec{S}_i \cdot \vec{S}_j + \lambda \sum_{\langle\langle i,j\rangle\rangle} \vec{S}_i \cdot \vec{S}_j - h\sum_i S_i^z$$







- Spin dimer /boson dictionary:

$$= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

  - Singlet : **empty** site
  - Triplet S^z=+1 : **occupied** by one boson
  - h=ext. mag field (||z) ⇔ boson chemical potential
- Strong enough ext. field: **Bose condensation** (of magnons)

Review: Giamarchi *et al.* 2008

→Long-ranged magnetic order in the plane perp. to the external mag. field.

$$\langle b^+_{k_0} \rangle = \sqrt{Nn_c}\exp(i\varphi)$$

$$\text{but } b^+_i \approx S^+_i = \frac{1}{2}\left(S^x_i + iS^y_i\right)$$

φ = spin direction in the XY plane

ex: $TlCuCl_3$
$BaCuSi_2O_6$ (also known as "Han purple")
…

Order parameters

What is an order parameter ?

- idea: An order parameter is an observable which allows to detect if a symmetry is broken or not.

- T=0
  A local observable O is an order for the symmetry g if:
  <x|O|x>=0 when the symmetry is not broken (g|x>~|x>, up to a possible phase)
  <x|O|x>≠0 when the sym. is broken.
  O is local, or a sum of local terms;

- Remark: to get an observable which expectation value vanishes in any symmetric state, use:

$$O' = O - \frac{1}{|G|}\sum_{g\in G} g^{-1} O g$$

$$\text{if } \forall g \quad g|x\rangle \sim |x\rangle \text{ then } \langle x|O'|x\rangle = 0$$

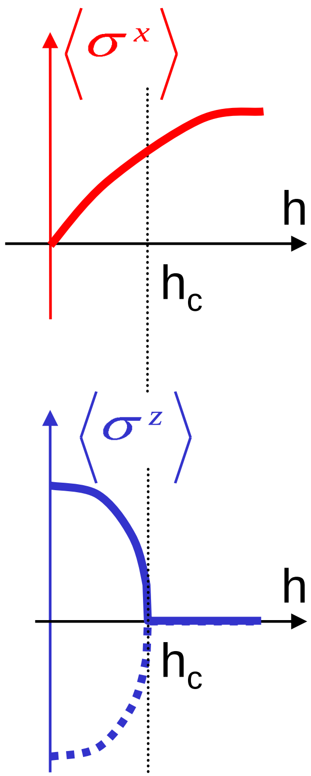
- T>0
  A local observable O is an order for the symmetry g if:
  <O> (thermal average) when the symmetry is not broken, and <O> can be non-zero when the sym. is broken.

Example of order parameters: quantum Ising model

- Ising model in transverse field

$$H = -\sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x \qquad \sigma_i^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_i^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- remarks: exactly solvable in 1d (spin chain, using Jordan-Wigner transf.)
  relevant to describe $LiHoF_4$ (then J_{ij}:= dipolar, long-ranged)
  $CsCoCl_3$, K_2CoF_4



- What is the symmetry group ?
  - Lattice symmetries (depends on Jij)
  - global spin flip: $\sigma^z \longrightarrow -\sigma^z$. Operator $g = \prod_{i=1}^{N} \sigma_i^x$
- Natural order parameter :
- Small h : $\langle m^z \rangle \neq 0$, large h: $\langle m^z \rangle = 0$. $\quad m^z = \frac{1}{N} \sum_i \sigma_i^z$
- Exercise: show that $g|x\rangle \sim |x\rangle$ implies $\langle x|\sigma^z|x\rangle = 0$
- Is σ^x also an order parameter ? No !

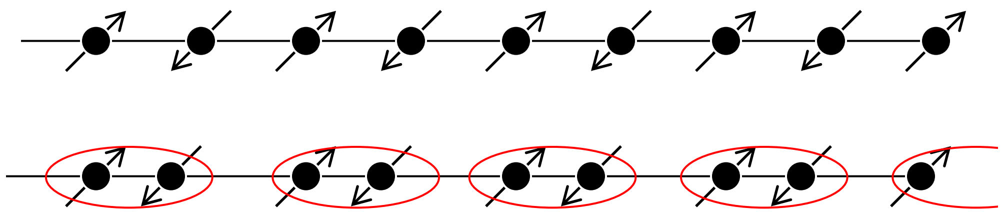
Spin Peierls

- Quantum spins coupled to an « elastic » lattice

$$H = \sum_{\langle ij \rangle} J(\vec{r}_i - \vec{r}_j)\vec{S}_i \cdot \vec{S}_j + \sum_{\langle ij \rangle} V(\vec{r}_i - \vec{r}_j)$$

- Spontaneous « dimerization »

 (magnetic energy gain > elastic energy cost)



- Examples of order parameters (translation symmetry breaking)

$$\sum_i (-1)^i \vec{S}_i \cdot \vec{S}_{i+1}$$

$$\sum_i (-1)^i \left| \vec{r}_i - \vec{r}_{i+1} \right|$$

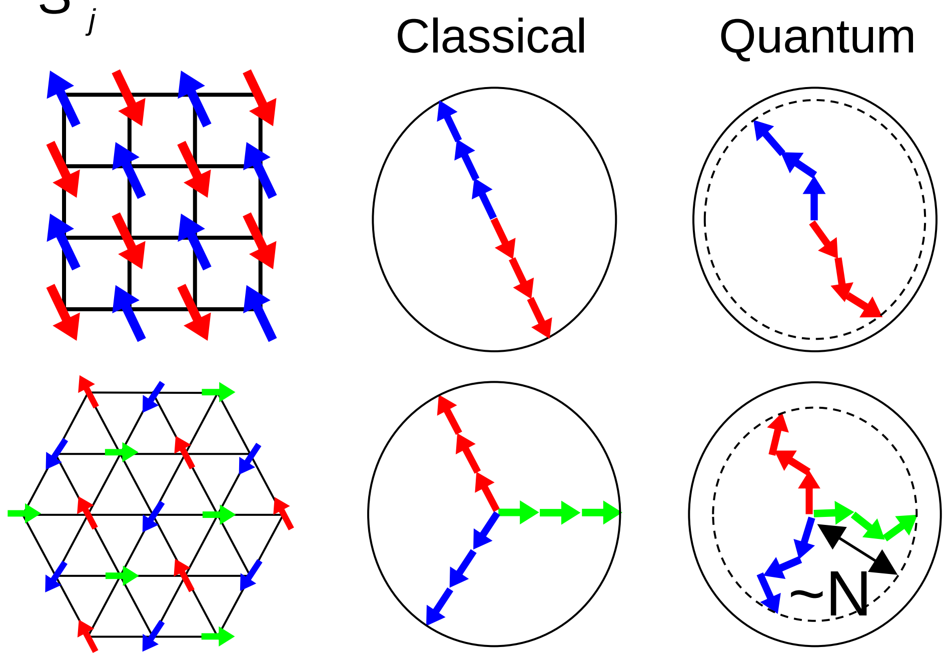
Example: $CuGeO_3$

- Dimerized phase: spin gap Δ for magnetic excitations.

 Is Δ an order parameter ?

Néel (antiferromagnetic) orders

$$H = \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$



$$\vec{S}(\mathbf{q}) = \sum_i e^{i\mathbf{q}\cdot\mathbf{r}_i} \vec{S}_i \quad \text{sublattice magnetization}$$

- Examples of order parameters which detect rotation and translation symmetry breakings.

$$\mathbf{q} = (\pi, \pi) \text{ square lattice}$$

$$\mathbf{q} = \left(\frac{4\pi}{3}, 0\right) \text{ triangular lattice}$$

Long-range order, correlation functions & susceptibilities

- **Spontaneous symmetry breaking ⇔ $\langle O_r O_{r'} \rangle$ is long-ranged**
- Take a large but *finite* system.

How can we measure if we are in the ordered or disordered phase ?

Problem $\langle O \rangle = 0$ in both phases (since the system is *finite*).

Solution: Compute $\langle O_r O_{r'} \rangle$ for sufficiently distant spins

If it does not decay to zero at large distances → broken symmetry phase.

- Structure factor:

$O=\Sigma_r O_r$, $O^2= \Sigma_{rr'} O_r O_{r'}$ $\langle O^2 \rangle = N \Sigma_r \langle O_0 O_r \rangle$ LRO ⇔ $\langle O^2 \rangle \sim N^2$

If $O=S(q)$, $\langle O^2 \rangle$ is accessible through neutron scattering for instance.

$|S(q)| \sim N^2$ gives Bragg peaks.

- One can also look at the susceptibility $H \rightarrow H(\lambda)=H - \lambda.O$

  $\chi = d\langle O \rangle / d\lambda$ (taken at $\lambda=0$) $= \langle O^2 \rangle / T$

- χ diverges as N^2 ⇔ LRO
- Remark: one can also define $\chi=[\langle O^2 \rangle - \langle O \rangle^2]/T$, in which case

χ is *finite* in both phases, and only diverges *at* the transition.

(a little bit of) Group theory

- Symmetry group G (finite for simplicity)
- An observable O
- One can generate other observables by acting with the symmetry operations

$$g \in G \qquad O_g = g^{-1} O g$$

- Chose a basis of the space (of observables) generated by {g^{-1} O g} : $\vec{O} = \begin{bmatrix} O_1 \\ \vdots \\ O_n \end{bmatrix}$

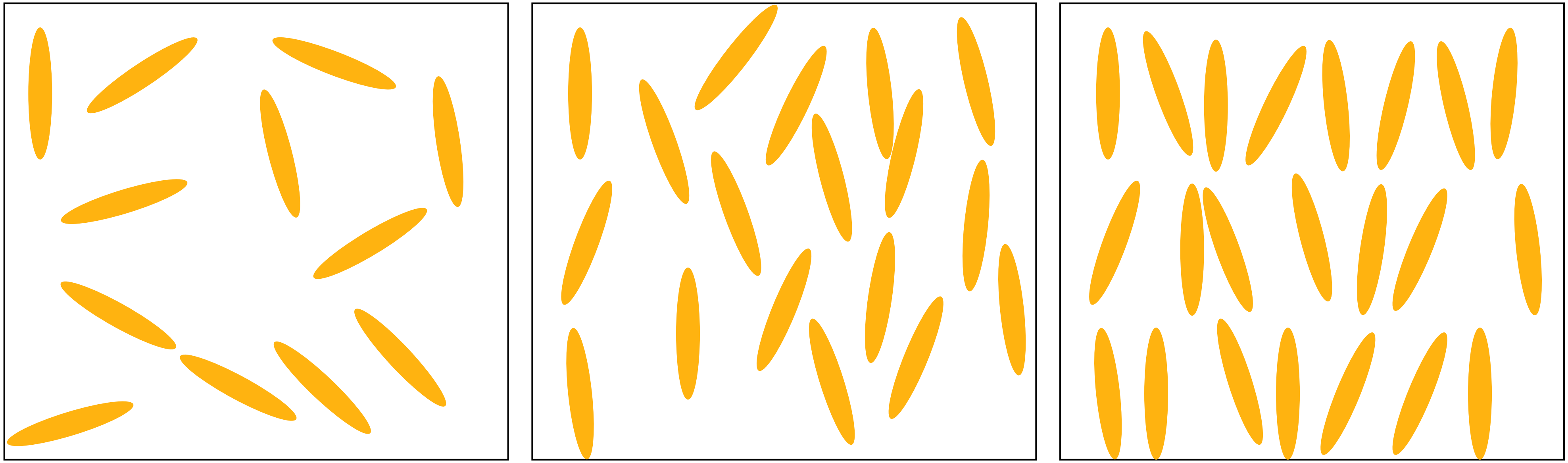
Ex. : $m^z = \sum_i S_i^z$ Rotations $\rightarrow \vec{m} = \begin{bmatrix} m^x \\ m^y \\ m^z \end{bmatrix}$

- This defines a **representation** of the group G
- Definition: a representation of a group G is an application which **associates an n*n invertible** (unitary) **matrix M(g) to each group element g**, with the property: M(g) * M(g') = M(gg') and M(Id)=identity matrix

- Decompose each g^{-1} O$_i$ g^{-1} in this basis : $g^{-1} O_i g = \sum_{j=1}^{n} M_{ij}(g) O_j$

The matrices M(g) form a rep. of the group G.

Nematic orders



Isotropic

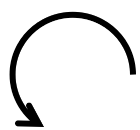
-

-

Nematic

exp(2iθ)

Broken sym.:



Smectic A

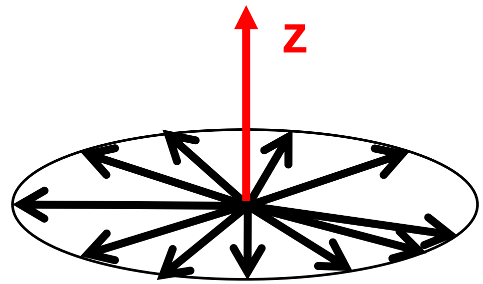
exp(2iθ)

exp(i k. r^y)

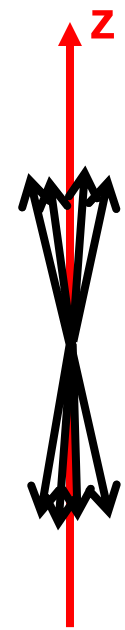
Broken sym.:

Example of order parameter: spin nematics

- A spin system in which the spins spontaneously chose a common plane, but no particular direction in this plane



- Or, selection of an axis, but no direction along that axis:



- Several quantum spin models are known to realize such kind of spin nematic phases

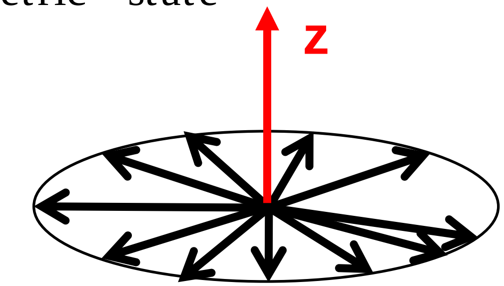
Lauchli *et al.* 2005; Shannon *et al.* 2006

- Experimental realization ? Perhaps $NiGa_2S_4$ (Nakatsuji *et al.* 2005) ?

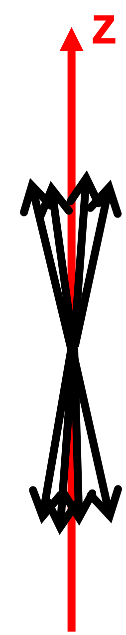
Example of order parameter: spin nematics

$$Q^1 \approx \sum_i (S_i^z)^2 \text{ ? no ! } \langle x|(S_i^z)^2|x\rangle = \frac{1}{3}\vec{S}_i^{\,2} = \frac{1}{3}S(S+1) \text{ in a symmetric state}$$

$$Q^1 \approx \sum_i \left[(S_i^z)^2 - \frac{1}{3}\left((S_i^x)^2 + (S_i^y)^2 + (S_i^z)^2 \right) \right]$$



$$\text{rotations} \rightarrow \vec{Q} = \begin{bmatrix} \frac{1}{\sqrt{3}}\left(2(S^z)^2 - (S^x)^2 - (S^y)^2\right) \\ (S^x)^2 - (S^y)^2 \\ 2S^xS^y \\ 2S^xS^z \\ 2S^yS^z \end{bmatrix}$$



=5 components of a rank-2 symmetric & traceless tensor

$Q^{ab}=S^aS^b-\delta^{ab}1/3[\ (S^x)^2+(S^y)^2+(S^z)^2\ ]$

= spin-2 irreducible representation of SO(3)

Ground state degeneracy & order parameters

- Phase with **discrete** broken symmetry → finite number of "ground-sates" |1>, |2>, … , |d>

- |1>,… , |d> form a representation **Γ** (of dim=d) of the symmetry group

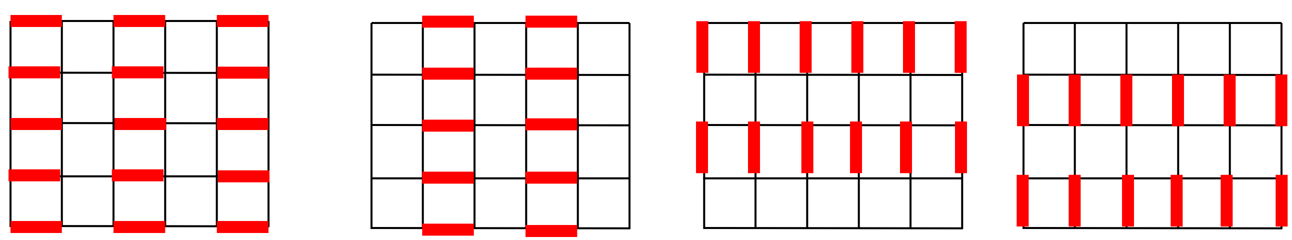
- **Γ** can be decomposed onto I.R. $\Gamma = 1 \oplus \gamma_a \oplus \gamma_b \oplus \gamma_c \oplus \ldots$

- One can find an order parameter associated to each of the **γ** above

(except the trivial one).

- Example: dimer on the square lattice & the columnar phase.

- Four ground states => **Γ** is a rep. of dim=4
- Decomposition over IR. $\Gamma_{\dim=4} = 1_{\dim=1} \oplus \gamma_{\dim=1} \oplus \gamma_{\dim=2}$
- Find 2 "irreducible" order parameters of dim=1 and dim=2 ? Exercise !



Landau theory of phase transitions (in a nutshell)

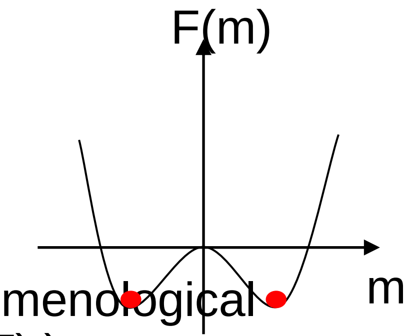
- Idea: to describe the "universal" (long-distance & low-energy) properties of a system in the vicinity of a phase transition, one does not need to know the behavior of all the particles… Instead, one only needs to consider a few macroscopic variables: the order parameter(s) of the competing phases.

- Expand the free energy in powers of the expectation values of the order parameters. At a given order, include all possible terms allowed by symmetries.

  ex:

  Symmetry: $m \leftrightarrow -m$

  $F(T,m)= \cancel{a(T)*m} + b(T)*m^2 + \cancel{c(T)*m^3}+d(T)*m^4$



- Minimize the free energy F(T,m) as a function of the phenomenological parameters (appearing in the expansion above: b(T) and d(T))

  (⇔mean field).

- Include space derivatives & fluctuations → better description of transitions
- Remark: in the group-theory language, "allowed by symmetry" means "component in the trivial representation". Useful when looking for "allowed" terms involving several (possibly complicated) order parameters.

Application of the Landau theory: cubic invariant

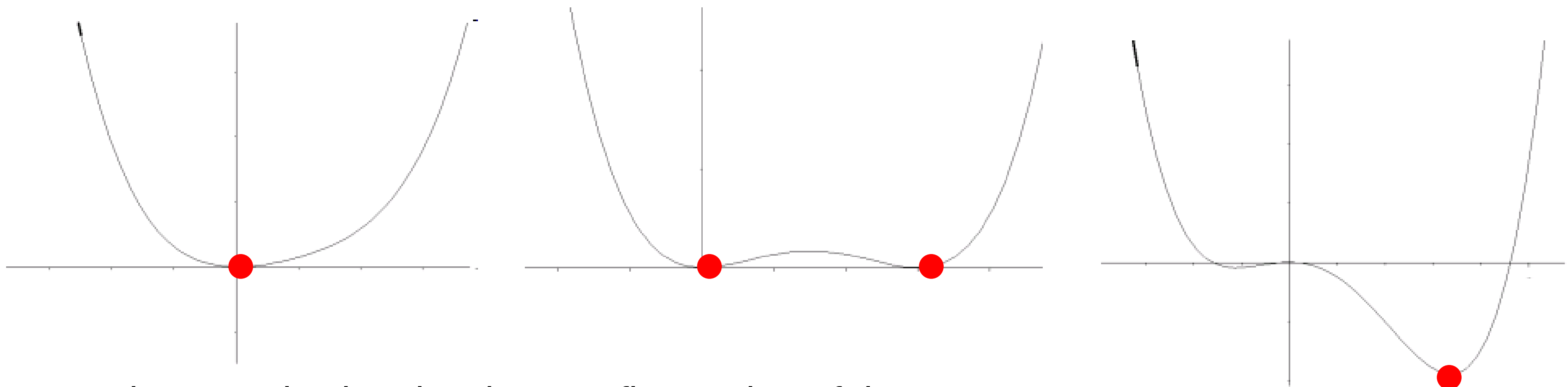
$$\vec{O} = \begin{bmatrix} O_1 \\ \vdots \\ O_n \end{bmatrix}$$

- n- component order parameter: $O^1 .. O^n$.
- Assume that some polynomial of degree 3 in the O^i is invariant under all the symmetries of the model.

Remark: Finding if such terms exist is easy using group theory the **characters** of representations !

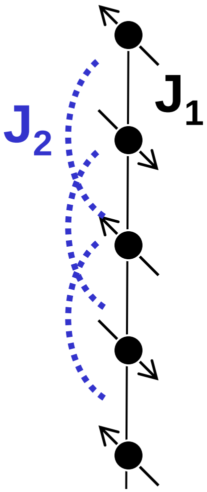
- Result: 1st order phase transition !

$F(O)=aO^2+bO^3+cO^4$



The generic situation (except fine tuning of the parameters) is a **jump from O=0 to O=finite**

Beyond Landau's theory of phase transitions

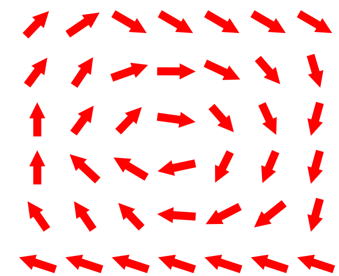


Sometimes, find order parameter(s) is not enough to describe phase transitions.

Examples:

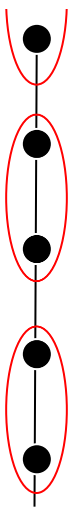
- **Liquid-gaz** transition
- **Metal-Insulator** transition
- 2d classical XY model and the "**Berezinsky-Kosterlitz-Thouless**" phase transition

  Low: T: algebraic spin-spin correlations   High T: exponential decay.

  ⇒In both phases: no spontaneously broken symmetry,
  and therefore no order parameter to distinguish the two phases.

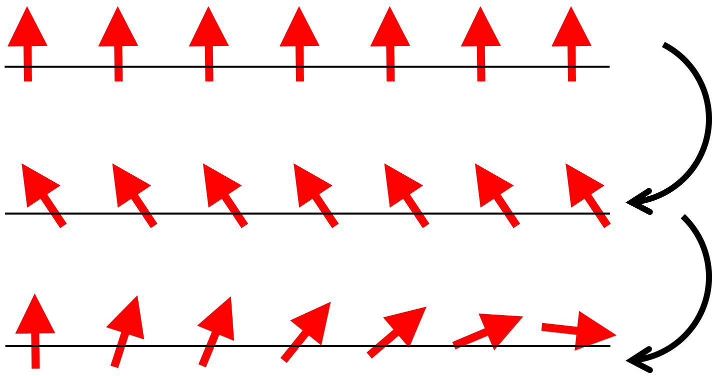
  Physics of **topological defects** (vortices) is not captured
  by a simple Landau approach.



- Transition between a dimerized and a gapless phase in the **J_1-J_2 Heisenberg chain** (spin=1/2).

  Even though the dimerized phase has a broken symmetry, it is in fact, same universality class as the BKT transition above.
- **Deconfined critical points** (Senthil *et al.* 2004): order parameters are there, but they are not the correct variables to describe the 2nd order quantum phase transitions in some particular 2d quantum magnets (Landau would predict them to be first order).

Continuous symmetry breaking & Nambu-Goldstone mode



Uniform rotation:
costs *nothing*

Long wavelength modulation
Of the rotation angle:
costs *little*

- Spontaneously broken **continuous** (global) symmetry +short-range interactions

⇒ **Gapless** (long-wavelength) excitations,

⇒ **linear dispersion relation**: $\omega(k) \sim k$.

NB: As many modes as broken symmetry generators.

- Examples:
  - spin waves in antiferromagnets (exercise: how many modes for a collinear magnet ? For a non-collinear magnet ?)
  - spin nematics
  - Sound in crystals
  - Sound in superfluity He^4, …
  - What about superconductors ? → Higgs mechanism

Mermin Wagner theorem

Hohenberg 1967; Mermin & Wagner 1966

- Spontaneous break down of a continuous symmetry is **forbidden** in the following situations :
  - Classical 1d and 2d, T>0
  - Quantum 1d T=0 (what about ferromagnets ?)

- Idea: Otherwise the thermally (quantum mechanically) excited Goldstone modes would destroy the long range order. Proof: See, for instance, Auerbach *"Interacting electrons & quantum magnetism"*, Springer 1994

- Absence of cont. sym. breaking does not mean no phase transition.
  Examples:
  - BKT in the 2d XY model: none of the two phase break any sym.
  - J_1-J_2 Heisenberg model on the square lattice: break down of a discrete lattice symmetry in the ordered phase. Continuous sym. are preserved. Weber *et al.* 2003

- 2d, T>0: No sym breaking, but correlation length can be huge: $\xi(T) \approx \exp(-T_0/T)$
- 3d couplings are often present…

Gauge invariance – « local symmetry »

Charged particle of mass m and charge q in presence of a vector potential A :

$$H = \frac{1}{2\mathrm{m}} \left(i\hbar \vec{\nabla} + q\vec{A} \right)^2$$

$$E = \langle \psi | H | \psi \rangle = \frac{1}{2\mathrm{m}} \int d^3 r \left| \left(i\hbar \vec{\nabla} + q\vec{A} \right) \psi(r) \right|^2$$

Gauge transformation :

"redundancy"

$$\psi(r) \to e^{i\Lambda(r)} \psi(r)$$

$$\vec{A} \to \vec{A} + \frac{i\hbar}{q} \vec{\nabla} \Lambda$$

Operator which implements the transformation : $g_\Lambda = \exp\left[i \frac{\Lambda(r)}{q} \left(q\, n(r) - div\, \vec{E} \right) \right]$

Generator of an « infinitesimal » gauge transformation: $G(r) = q\, n(r) - div\, \vec{E}$

Gauss Law: (ρ(r)-divE)|Phys>=0

⇔ **physical states must be invariant under gauge transformations.**

→ Avoid having several spurious (gauge equivalent) states for the same "physical" state.

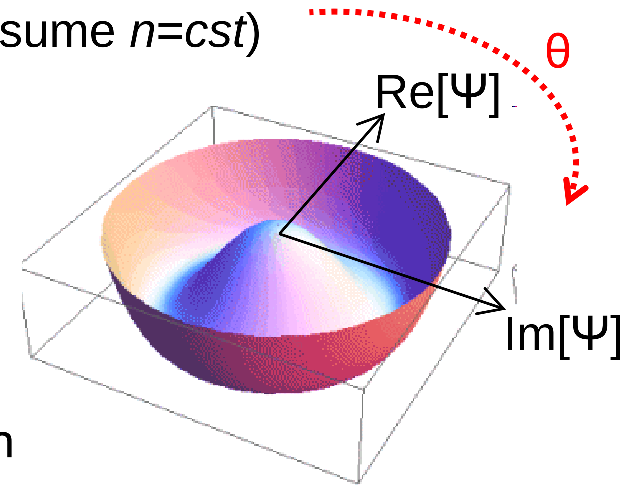
Anderson-Higgs mechanism (Meissner effect)

Particle with mass m and charge q:

$$E = \frac{1}{2m} \int d^3 r \left| \left(i\hbar \vec{\nabla} + q\vec{A} \right) \psi(r) \right|^2$$

But also, ψ(r): wave-function of a Bose-Einstein condensate (assume *n=cst*)

$$\psi(r) = \sqrt{n} e^{i\theta(r)}$$



One can **choose a gauge in which θ=0 everywhere**

(→no phase degree of freedom anymore, no Goldtsone anymore)

$$E = \frac{q^2 n}{2m} \int d^3 r \left| \vec{A} \right|^2$$

="**mass term**" for the photon

→finite excitation gap for the electromagnetic field

Higgs mechanism:

the Goldstone mode is "eaten up" by the gauge boson, which acquires a gap.

- Superconductivity & *Meissner effect*
- Effective theories for strongly correlated systems are often *gauge theories*.
- Particle physics & electroweak symmetry breaking (~200 GeV). Higgs, W & Z bosons.

Conclusions

- Symmetries and broken symmetries are important !
  and interesting, and useful, …. ☺
  - Starting point to define/distinguish states of matter
  - Understanding some low-energy degrees of freedom (Goldstone etc.)
  - Description/prediction of phase transitions (Landau theory)

- Some phases and phase transitions require however to go beyond Landau's description in terms of broken symmetry. Several active fields of research :
  - quantum Hall effect
  - spin liquids (in frustrated magnets)
  - topological insulators
  - Deconfined critical points
  - Confinement / deconfinement